

**EG02041 Advanced Mathematics**  
**Class 6: First-Order Differential Equations**

### 8.2 Linear First-Order Differential Equations

**Definition:** A first order differential equation is called linear if it can be written in the form:

$$y' + p(x) \cdot y = q(x)$$

If  $q(x) = 0$ , then we say that the differential equation is *homogeneous*. Otherwise we say that differential equation is *nonhomogeneous*.

For an example,

$$(x^2 + 1)y' - xy = e^x$$

May be written as a linear equation, divided by  $(x^2+1)$ :

$$y' - \frac{x}{x^2 + 1}y = \frac{e^x}{x^2 + 1}$$

$$p(x) = \frac{-x}{(x^2+1)} \quad \text{and} \quad q(x) = \frac{e^x}{(x^2+1)}$$

**Theorem #2:** Consider the initial-value problem:

$$y' + p(x) \cdot y = q(x) \quad y(x_0) = y_0$$

Suppose that,  $p$  and  $q$  are both continuous on an interval  $I$  that contains  $x_0$ . Then there is an initial value problem defined for every point in  $I$ .

### Solving Linear Differential Equations

**Theorem #3:** The general solution for first order differential equation

$$y' + p(x) \cdot y = q(x)$$

May be found by solving:

$$[G(x) \cdot y]' = q(x) \cdot G(x)$$

Where  $G(x) = e^{F(x)}$  and  $F(x)$  is an antiderivative of  $p(x)$

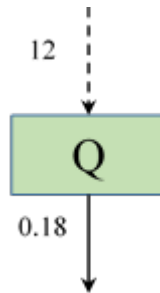
Using this theorem, we can solve any first order differential equation as long as we can compute the integral.

**Example 3:** solve  $y' - 3x^2y = x^2$ .

### One-compartment Models

**Example 5: Biohazards.** A mussel is placed into water polluted with polychlorinated biphenyls (PCBs). Let  $Q(t)$  be the concentration of PCB in the mussel in micrograms (per gram of tissue) after  $t$  days. For low

concentrations of pollution, the mussel absorbs PCBs at the rate of 12 micrograms of PCB per gram of tissue per day. Also, the elimination rate of PCBs from the mussel is  $0.18Q$  micrograms per gram of tissue per day. Construct a differential equation for  $Q$ .



**Example 6: Biohazards.** Let  $Q(t)$  be as in **Example 5**.

- Solve for  $Q$  if no PCB is initially present in the mussel.
- For large values of  $t$ , what value does  $Q(t)$  approach?

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### Relative, Uptake, and Downtake Rates

**Example 1: Mixing Chemicals.** A tank contains 100 gal of brine whose concentration is 2.5 lb of salt per gal. Brine containing 2 lb of salt per gal runs into the tank at a rate of 5 gal per min, and the mixture runs out of the tank at the same rate. The concentration of salt within the tank is kept uniform by stirring. Let  $S(t)$  be the amount of salt in the tank in pounds at time  $t$ .

- Construct a one-compartment model for  $S$ .
- Construct a differential equation for  $S$ .
- Determine how much salt is in the tank at time  $t$ .
- For large  $t$ , how much salt is in the tank?

